

: Careful synchronization of partial automata

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Complete automata

First consider a complete DFA: $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$.

- Q is the state set
- Σ is the input alphabet
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function

Here δ is supposed to be everywhere defined.

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Synchronization of complete automata

Definition

A DFA \mathcal{A} is called **synchronizing** if there exists a word $w \in \Sigma^*$ whose action resets \mathcal{A} , that is, leaves the automaton in one particular state no matter which state in Q it started at:
 $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

In symbols, $|\delta(Q, w)| = 1$. Here $\delta(Q, w) = \{\delta(q, w) \mid q \in Q\}$.
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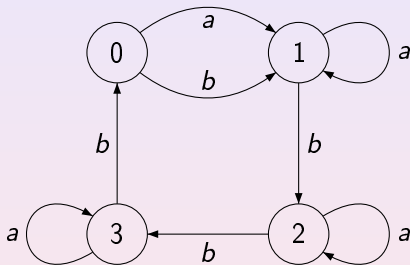
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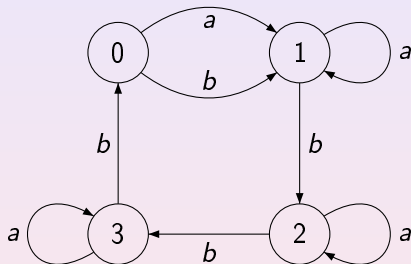
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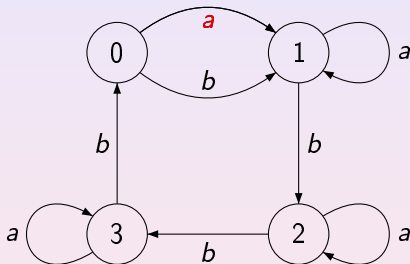
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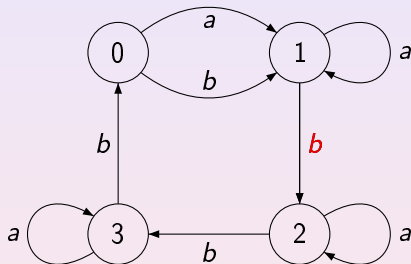
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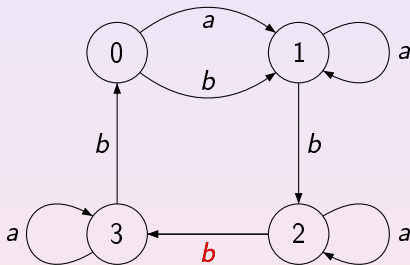
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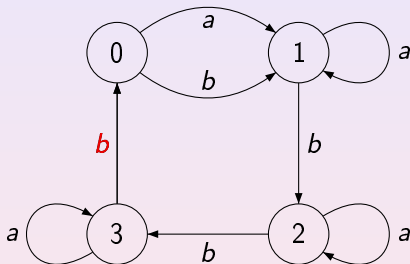
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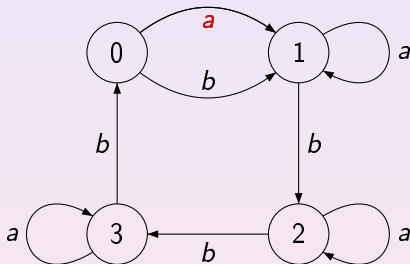
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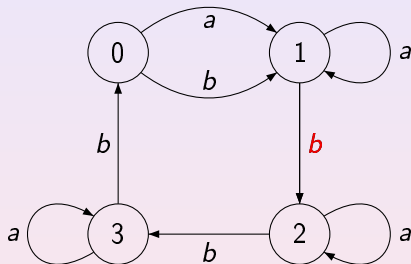
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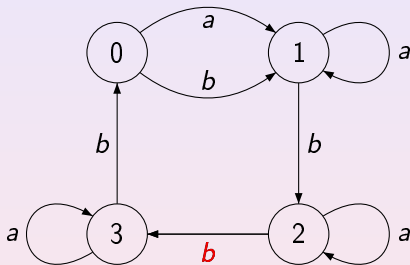
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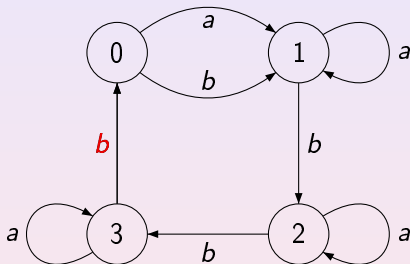
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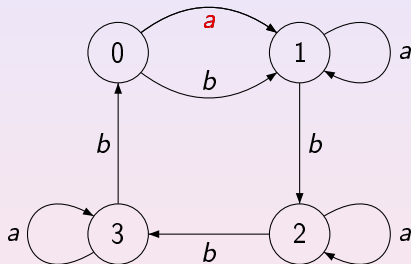
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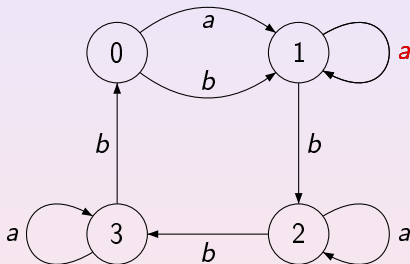
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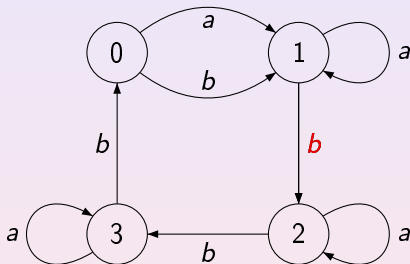
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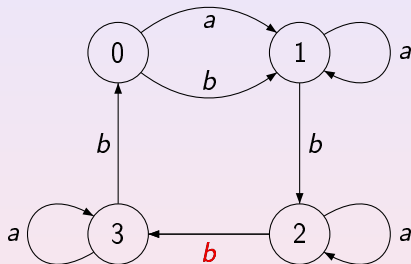
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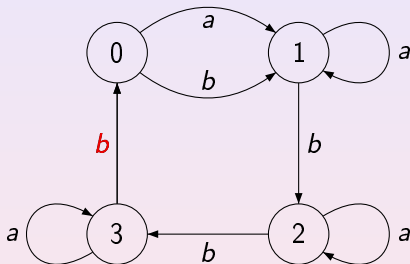
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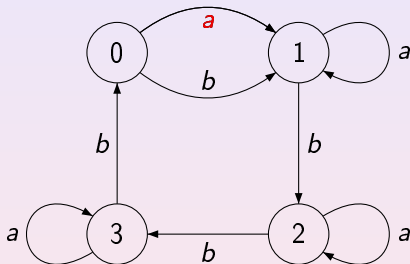
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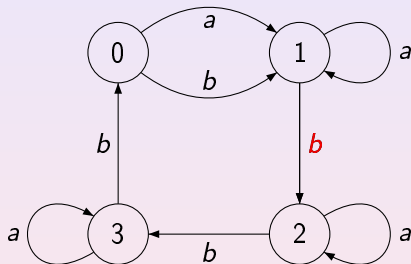
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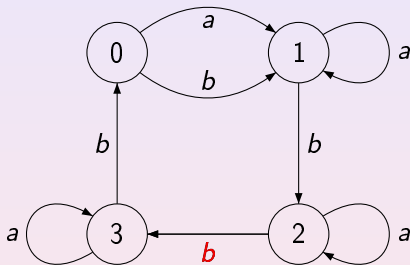
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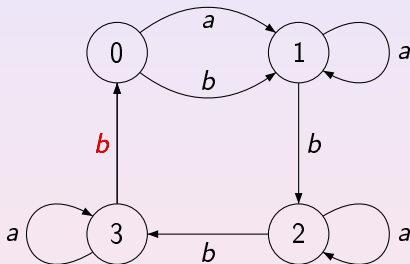
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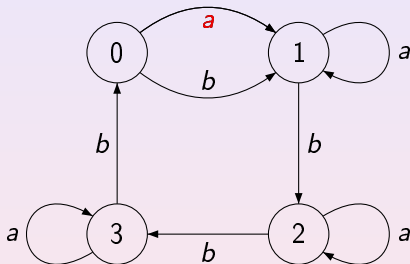
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The idea of synchronization is pretty natural and of obvious importance: we aim to restore control over a device whose current state is not known.

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Partial automata

Now consider a **partial** DFA: $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$.

The only (but essential!) difference is that now the transition function δ is allowed to be partial, that is $\delta(q, a)$ can be undefined for some $q \in Q$ and $a \in \Sigma$.

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Say that $w = a_1 \cdots a_\ell$ with $a_1, \dots, a_\ell \in \Sigma$ is a **careful reset word** for a partial DFA $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ if:

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Thus, w is an instruction that the automaton is always able to follow and that brings it to a state which is independent of the initial state.

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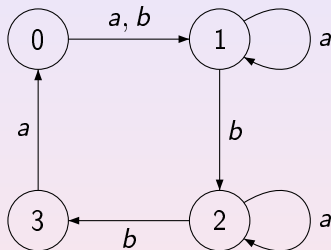
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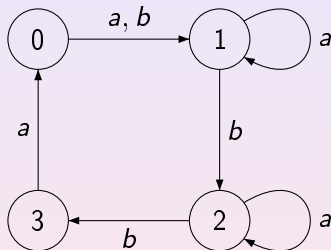
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Power automata

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For the complete case, a straightforward solution comes from the classic power automaton construction.

The **power automaton** $\mathcal{P}(\mathcal{A})$ of a complete DFA $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$:

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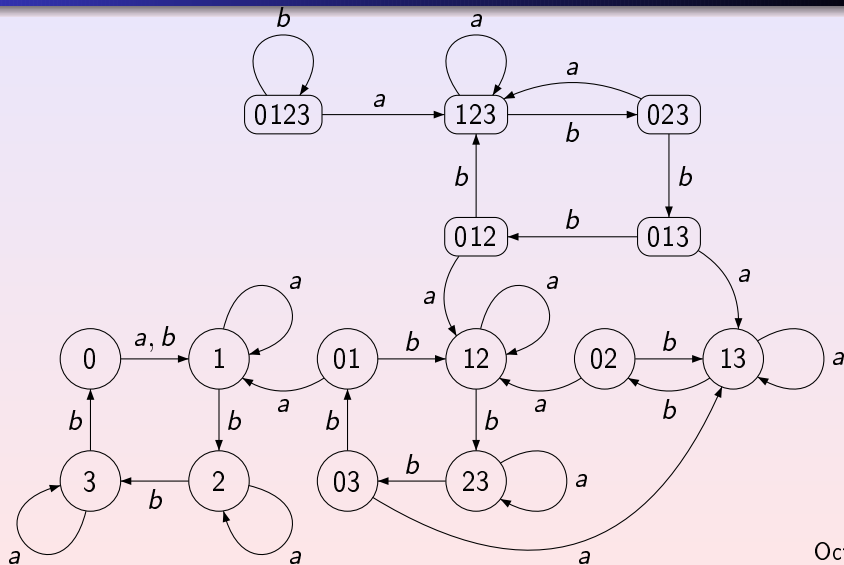
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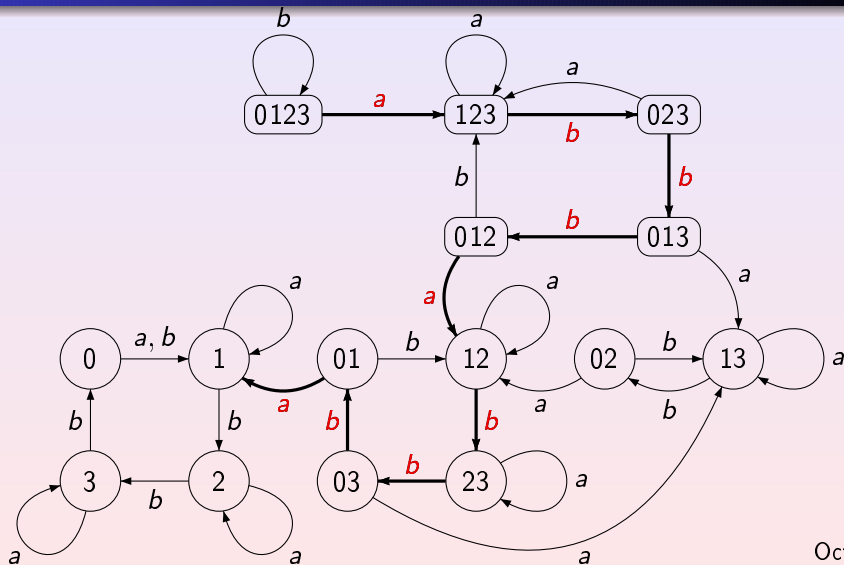
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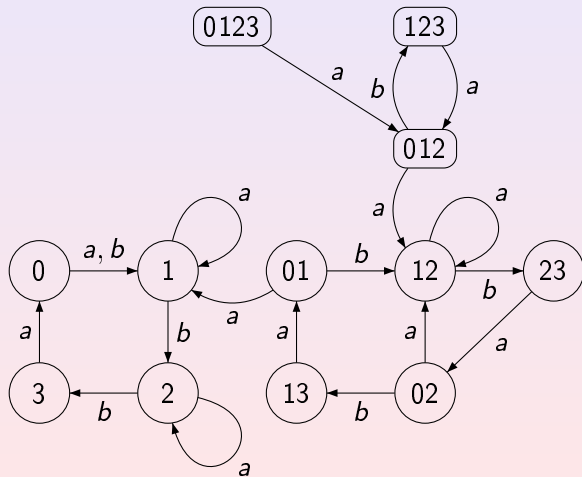
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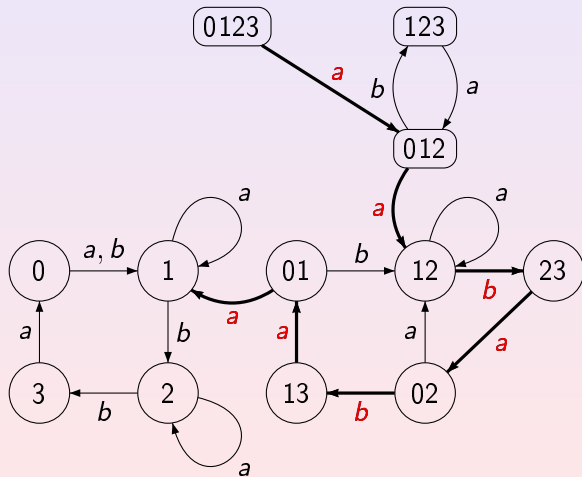
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Efficiency: complete case

Algorithms arising from the power set construction are exponential w.r.t. the size of \mathcal{A} .

For the complete case, the following result by Černý gives a polynomial algorithm:

Proposition (Černý, 1964)

A complete DFA $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ is synchronizing iff for every $q, q' \in Q$ there exists a word $w \in \Sigma^*$ such that $\delta(q, w) = \delta(q', w)$.

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Oct 3, 2008

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Upper bounds

The power automaton construction gives an upper bound for the length of the shortest reset word of a given synchronizing automaton $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ (complete or partial): if the automaton has n states, it has a reset word of length at most $2^n - n - 1$. (The shortest path in $\mathcal{P}^1(\mathcal{A})$ starting at Q and ending at a singleton does not visit any state twice and ends at the first singleton it reaches.)

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A complete synchronizing automaton with n states has a reset word of length at most $\frac{n^3 - n}{6}$.

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We do not know any reasonable upper bound for the partial case. It definitely cannot be polynomial because there exist **non-polynomial lower bounds**.

The first such bound has been found by Ito and Shikishima-Tsuji:

Theorem (Ito and Shikishima-Tsuji, 2004)

There exists a series of partial carefully synchronizing automata with n states and shortest careful reset words of length $O(2^{\frac{n}{2}})$.

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Open problems

Thus, we have the following open problems:

Upper Bound Problem

Does there exist a **subexponential** (in # of states) **upper bound** for the length of careful reset words of partial carefully synchronizing automata?

Lower Bound Problem

Does there exist an **exponential** (in # of states) **lower bound** for the length of careful reset words of partial carefully synchronizing automata?

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The Černý automata

Recall that the situation is not easy even for the complete case.

We know an upper bound: there always exists a reset word of length $\frac{n^3 - n}{6}$. What about a lower bound?

In his 1964 paper Černý constructed a series \mathcal{C}_n , $n = 2, 3, \dots$, of synchronizing automata over 2 letters. The states of \mathcal{C}_n are the residues modulo n , and the input letters a and b act as follows:

$$\delta(0, a) = 1, \delta(m, a) = m \text{ for } 0 < m < n, \delta(m, b) = m+1 \pmod{n}.$$

The automaton in our first example is \mathcal{C}_4 .

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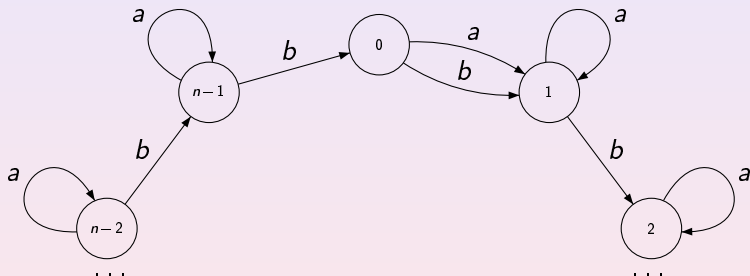
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Here is a generic automaton from the Černý series:

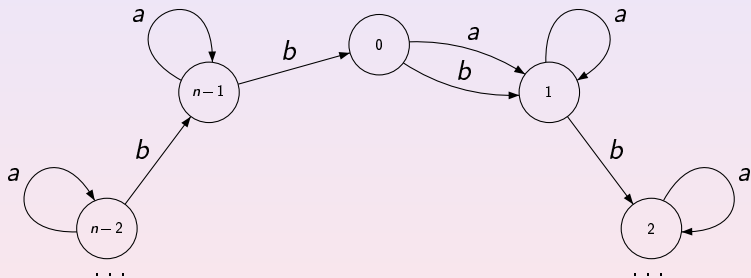


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Černý conjectured that the automata \mathcal{C}_n represent the worst possible case, that is, every complete synchronizing automata with n states admits a reset word of length at most $(n - 1)^2$.

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Strongly connected synchronizing automata

Assume that a complete synchronizing automaton is strongly connected as a digraph. Such an automaton can be reset to **any state**. That is, to every state q of the automaton one can assign an instruction (a reset word) w_q such that following w_q one will surely arrive at q from any initial state.

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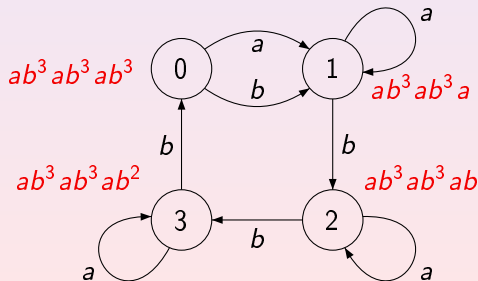
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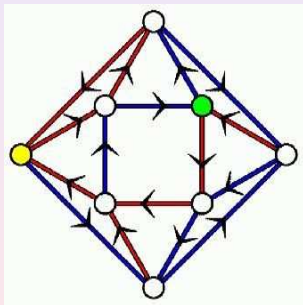
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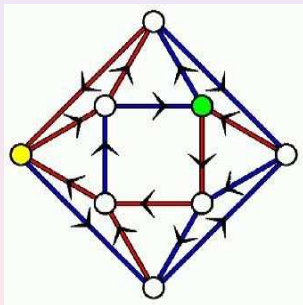


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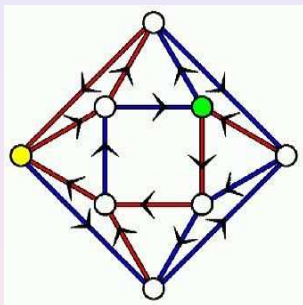
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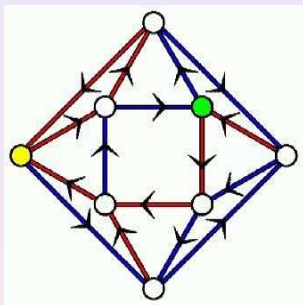


For the green node: blue-blue-red-blue-blue-red-blue-blue-red.

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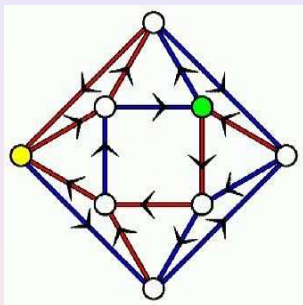


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Road coloring

Now suppose that we have a transport network, that is, a strongly connected digraph.

We aim to help people to orientate in it, and as we have seen, a neat solution may consist in coloring the roads such that our digraph becomes a synchronizing automaton. When is such a coloring possible?

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An obvious necessary condition: **all vertices should have the same out-degree.**

A less obvious necessary condition is called **aperiodicity** or **primitivity**:
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The name *Road Coloring Conjecture* was used for the claim that the two necessary conditions (constant out-degree and primitivity) are in fact sufficient.

In other words: every strongly connected primitive digraph with constant out-degree admits a synchronizing coloring.

The Road Coloring Conjecture was explicitly stated by Adler, Goodwyn and Weiss in 1977 (Equivalence of topological Markov shifts, Israel J. Math., 27, 49–63). In an implicit form it was present already in an earlier memoir by Adler and Weiss (Similarity of automorphisms of the torus, Memoirs Amer. Math. Soc., 98 (1970)) almost 40 years ago. Finally the problem was solved (in the affirmative) in August 2007 by Avraham Trahtman.

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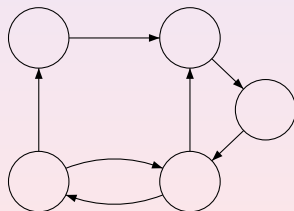
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