Synchronizing Aperiodic Automata

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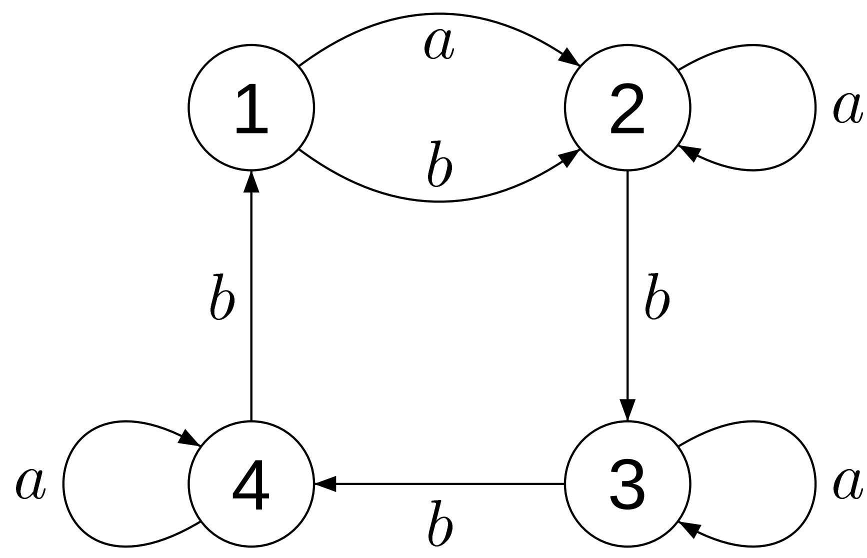
Any word w with this property is said to be a *reset word* for the automaton.

Synchronizing automata



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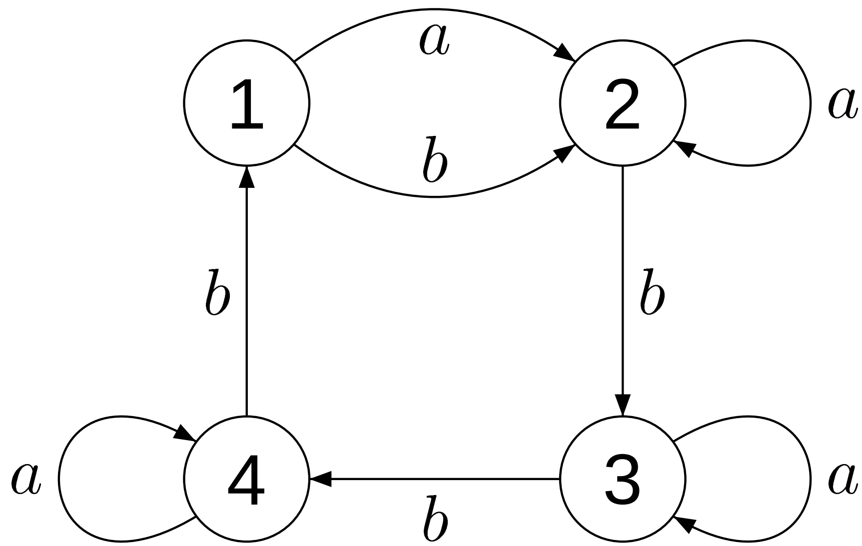

A reset sequence of actions is $abbbabbba$. Applying it at any state brings the automaton to the state 2.

Synchronizing Automata

The notion was formalized in 1964 in a paper by Jan Černý (Poznámka k homogénnym eksperimentom s konecnými automatami, Mat.-Fyz. Cas. Slovensk. Akad. Vied. 14 (1964) 208–216) though implicitly it had been studied since 1956.

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Think of a satellite which loops around the Moon and cannot be controlled from the Earth while "behind" the Moon (Černý's original motivation).

Engineering Applications

In the 80s, the notion was reinvented by engineers working in *robotics* or, more precisely, *robotic manipulation* which deals with part handling problems in industrial automation such as part feeding, fixturing, loading, assembly and packing (and which is therefore of utmost and direct practical importance).

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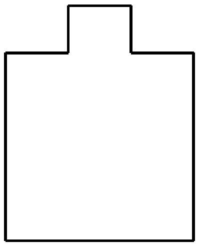

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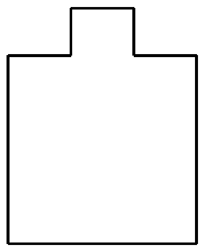

Such parts arrive at manufacturing sites in boxes and they need to be sorted and oriented before assembly.

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Assume that only four initial orientations of the part shown above are possible, namely, the following ones:

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Suppose that prior the assembly the part should take the “bump-left” orientation (the second one on the picture). Thus, one has to construct an orienter which action will put the part in the prescribed position independently of its initial orientation.

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We put parts to be oriented on a conveyer belt which takes them to the assembly point and let the stream of the details encounter a series of passive obstacles of two types (*high* and *low*) placed along the belt.

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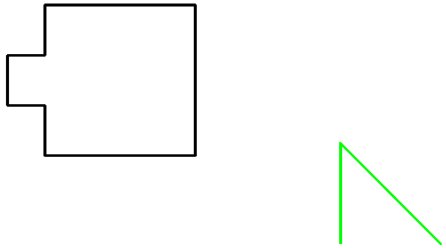
A high obstacle is high enough so that any part on the belt encounters this obstacle by its rightmost low angle.



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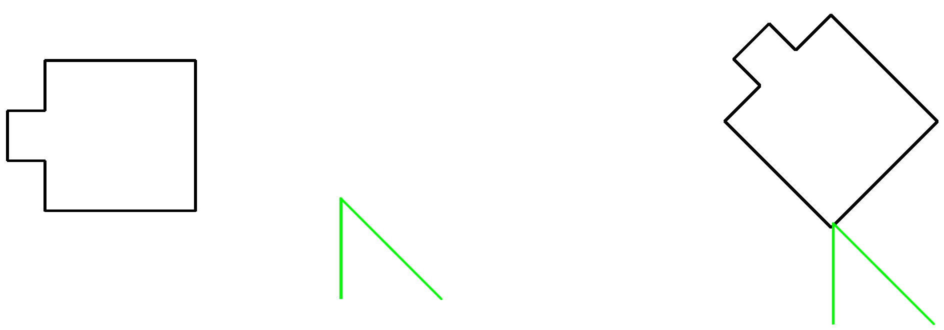

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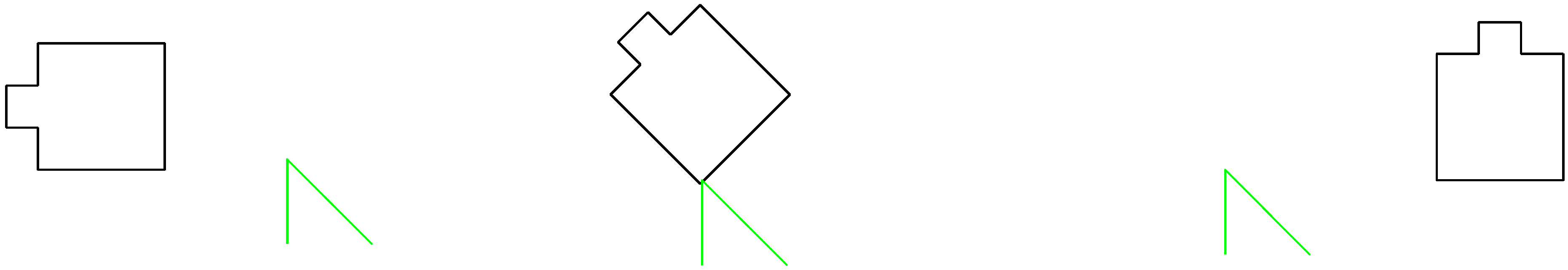

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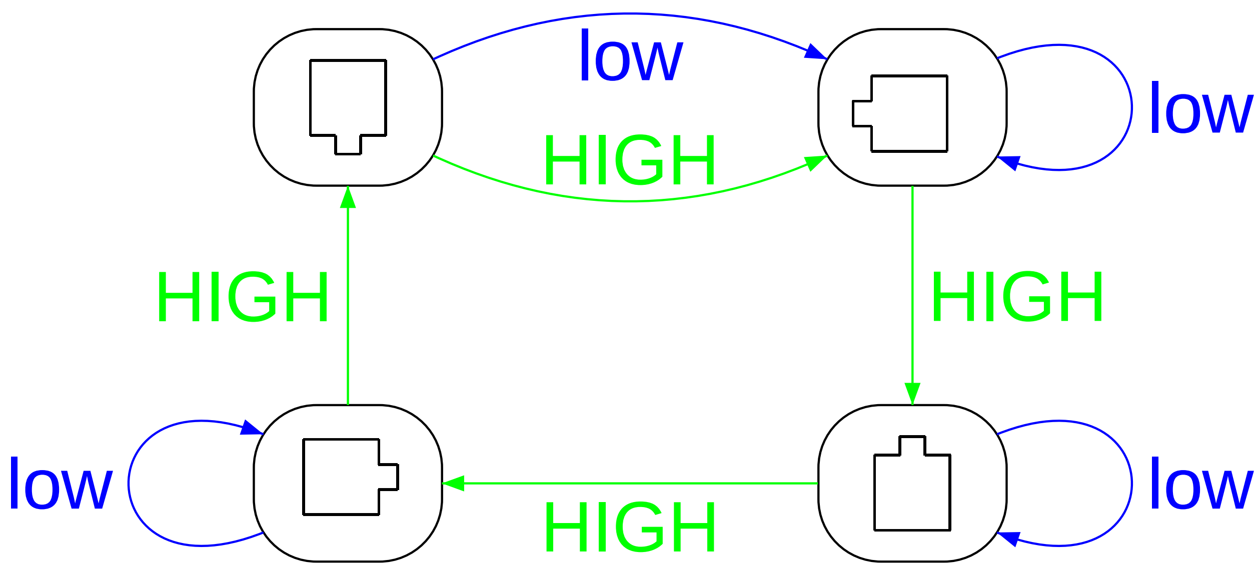
The following schema summarizes how the obstacles effect the orientation of the part in question:



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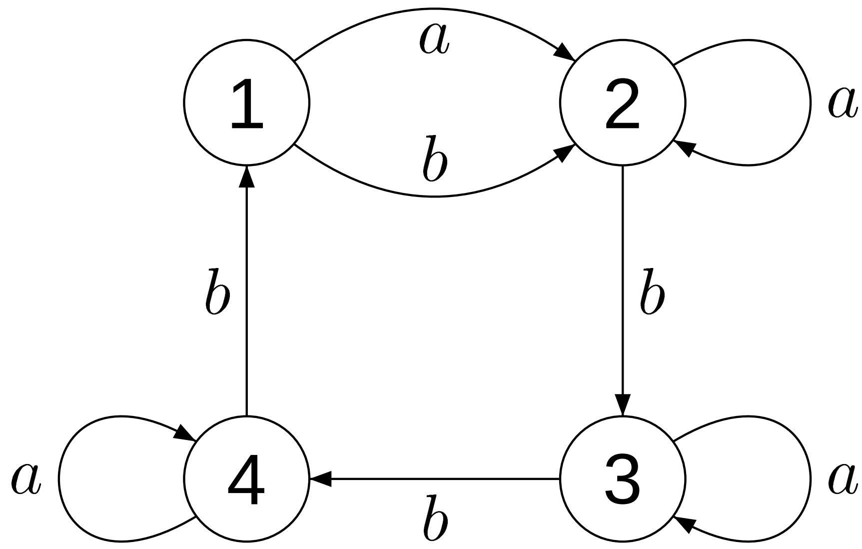

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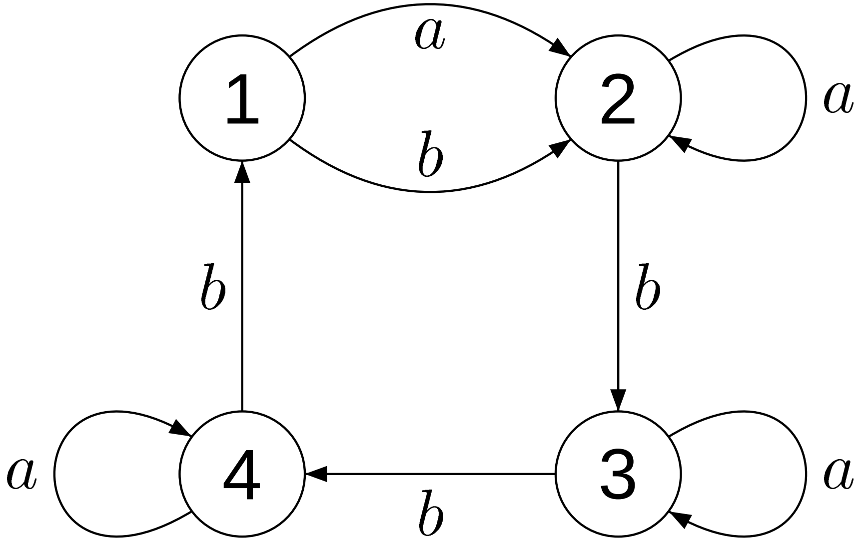

– this was our example of a synchronizing automaton, and we saw that $abbbabbba$ is a reset sequence of actions. Hence the series of obstacles

low-HIGH-HIGH-HIGH-low-HIGH-HIGH-HIGH-low

yields the desired sensorless orienter.

Further Applications

In *DNA-computing*, there is a fast progressing work by Ehud Shapiro's group on "*soup of automata*" (Programmable and autonomous computing machine made of biomolecules, Nature 414, no.1 (November 22, 2001) 430–434; DNA molecule provides a computing machine with both data and fuel, Proc. National Acad. Sci. USA 100 (2003) 2191–2196, etc).

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The simply looking conjecture is still open in general!!

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Given a DFA $\mathscr{A}$ and a positive integer ℓ, the problem whether or not $\mathscr{A}$ has a reset word of length $\leq \ell$ is NP-complete (D. Eppstein, 1990; P. Goralčik and V. Koubek, 1995; A. Salomaa, 2003).

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Given a DFA $\mathscr{A}$ and a positive integer ℓ, the problem whether or not the shortest reset word for $\mathscr{A}$ has length ℓ is co-NP-hard (W. Samotij, 2007).

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Synchronization issues remain difficult when restricted to $\mathbf{Ap}$.

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The gap between the upper and the lower bounds is rather drastic.

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The idea: consider certain properties that guarantee aperiodicity and are easier to check.

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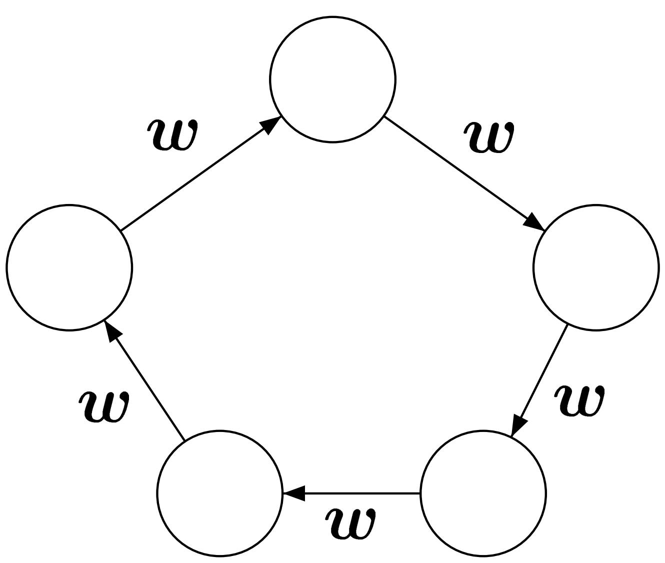

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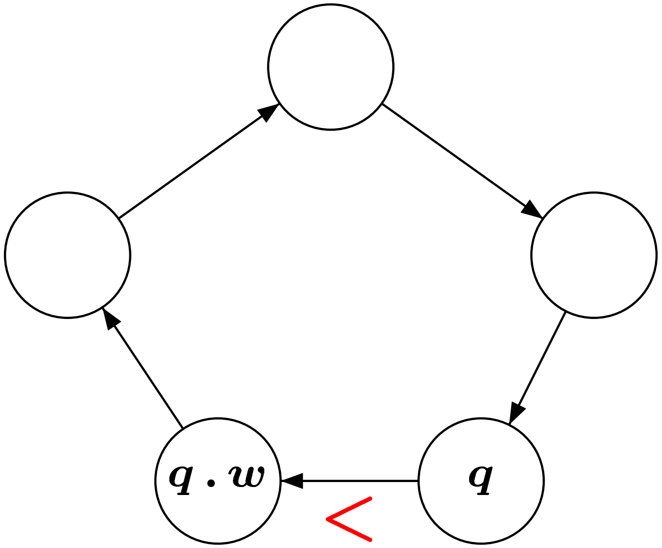

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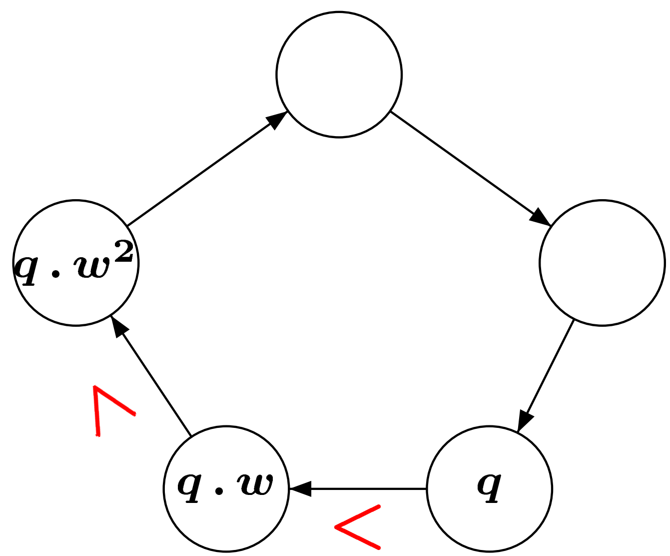

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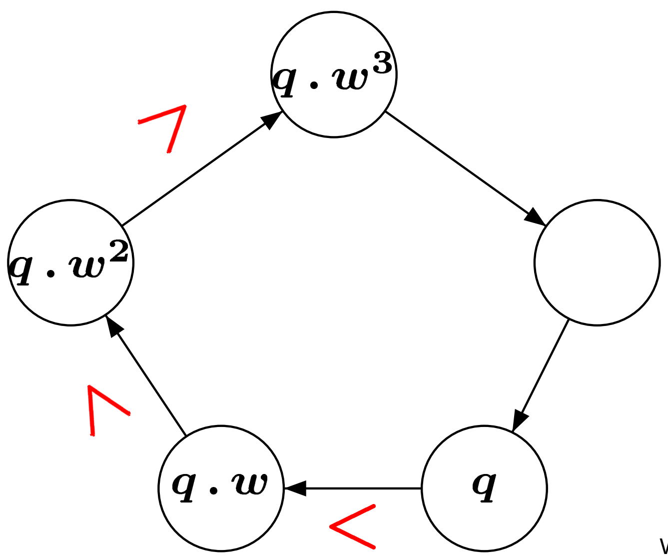

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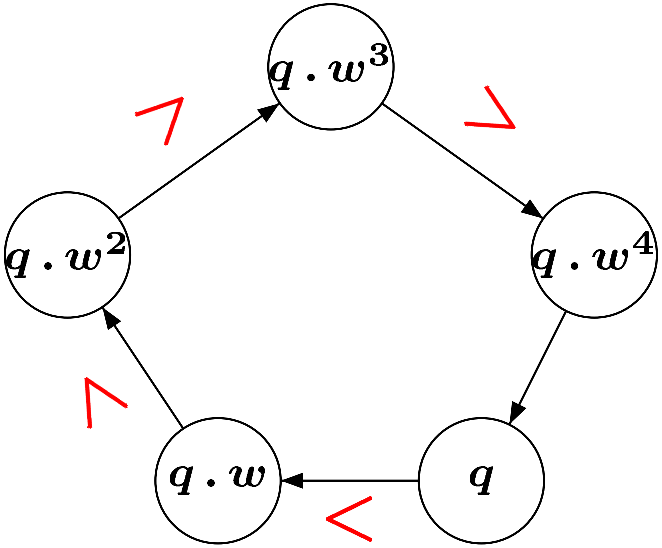

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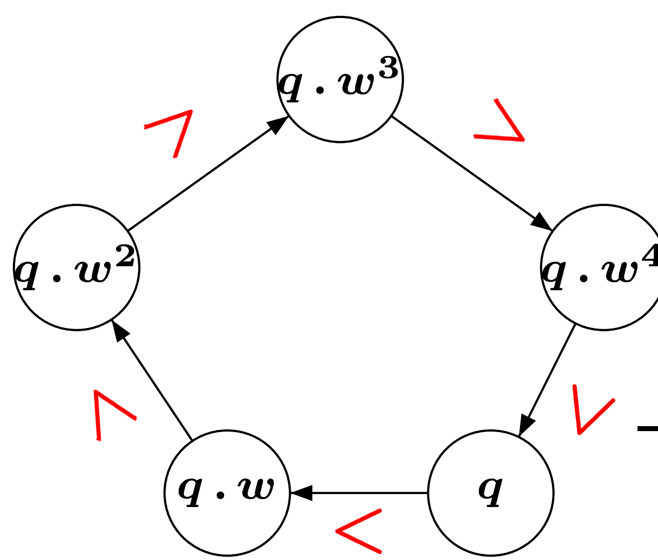

– contradiction!

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A much more difficult case of so-called 0-monotonic automata was analyzed by Ananichev in 2005. A DFA $\mathscr{A} = \langle \boldsymbol{Q}, \Sigma, \boldsymbol{\delta} \rangle$ is 0-*monotonic* if it has a unique *sink* $\boldsymbol{s} \in \boldsymbol{Q}$ and $\boldsymbol{Q} \setminus \{s\}$ admits a linear order $\leq$ preserved by the restrictions of the transformations $\boldsymbol{\delta}(\sqcup, \boldsymbol{a})$ to $\boldsymbol{Q} \setminus \{s\}$.

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A much more difficult case of so-called 0-monotonic automata was analyzed by Ananichev in 2005. A DFA $\mathscr{A} = \langle \boldsymbol{Q}, \boldsymbol{\Sigma}, \boldsymbol{\delta} \rangle$ is 0-*monotonic* if it has a unique *sink* $\boldsymbol{s} \in \boldsymbol{Q}$ and $\boldsymbol{Q} \setminus \{\boldsymbol{s}\}$ admits a linear order $\leq$ preserved by the restrictions of the transformations $\boldsymbol{\delta}(\sqcup, \boldsymbol{a})$ to $\boldsymbol{Q} \setminus \{\boldsymbol{s}\}$. Clearly, 0-monotonic automata are in a 1-1 correspondence with incomplete monotonic automata.

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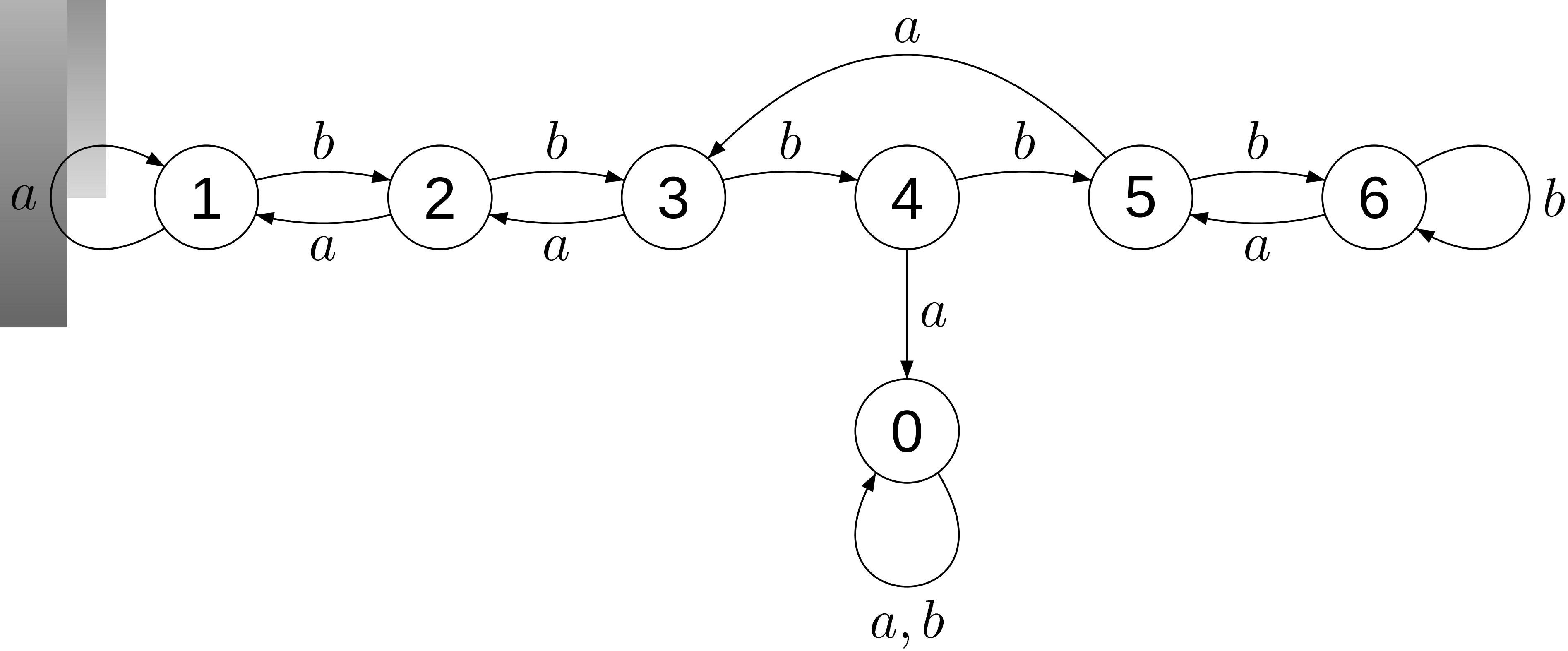

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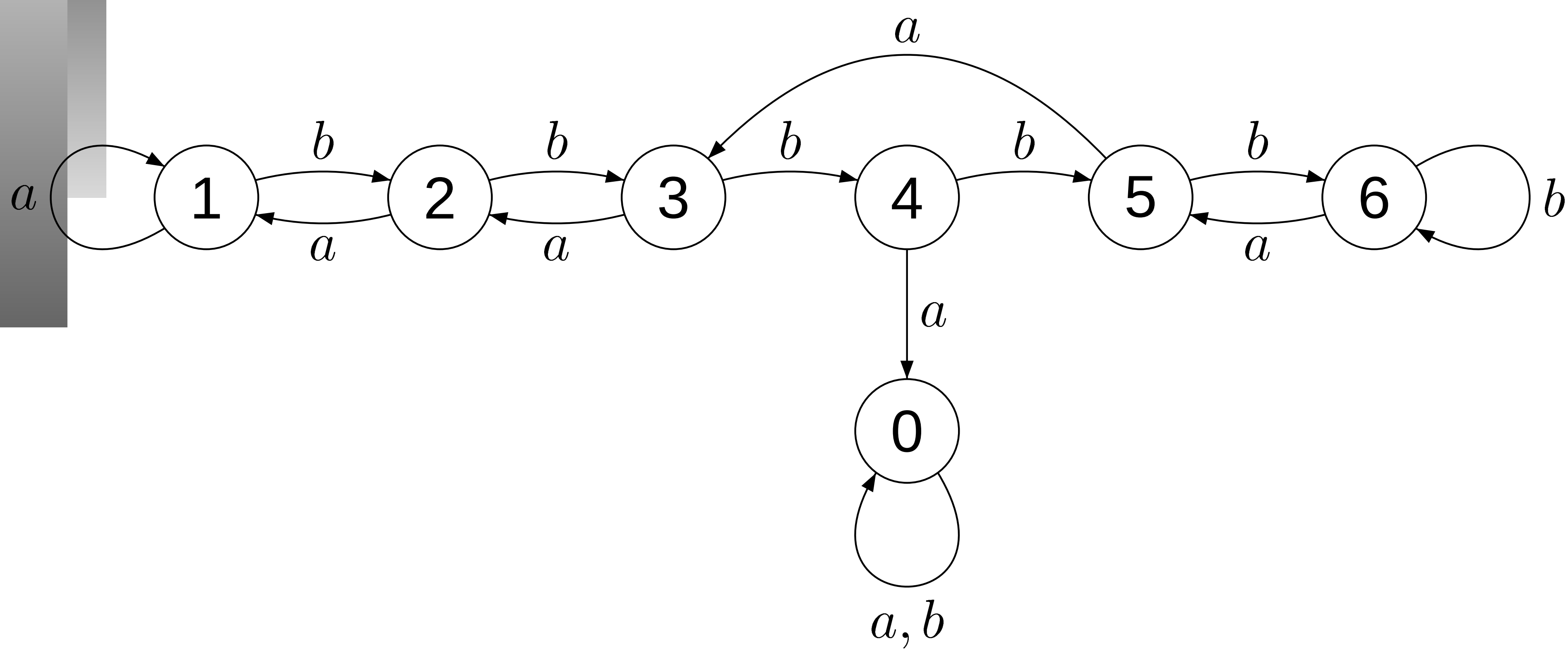

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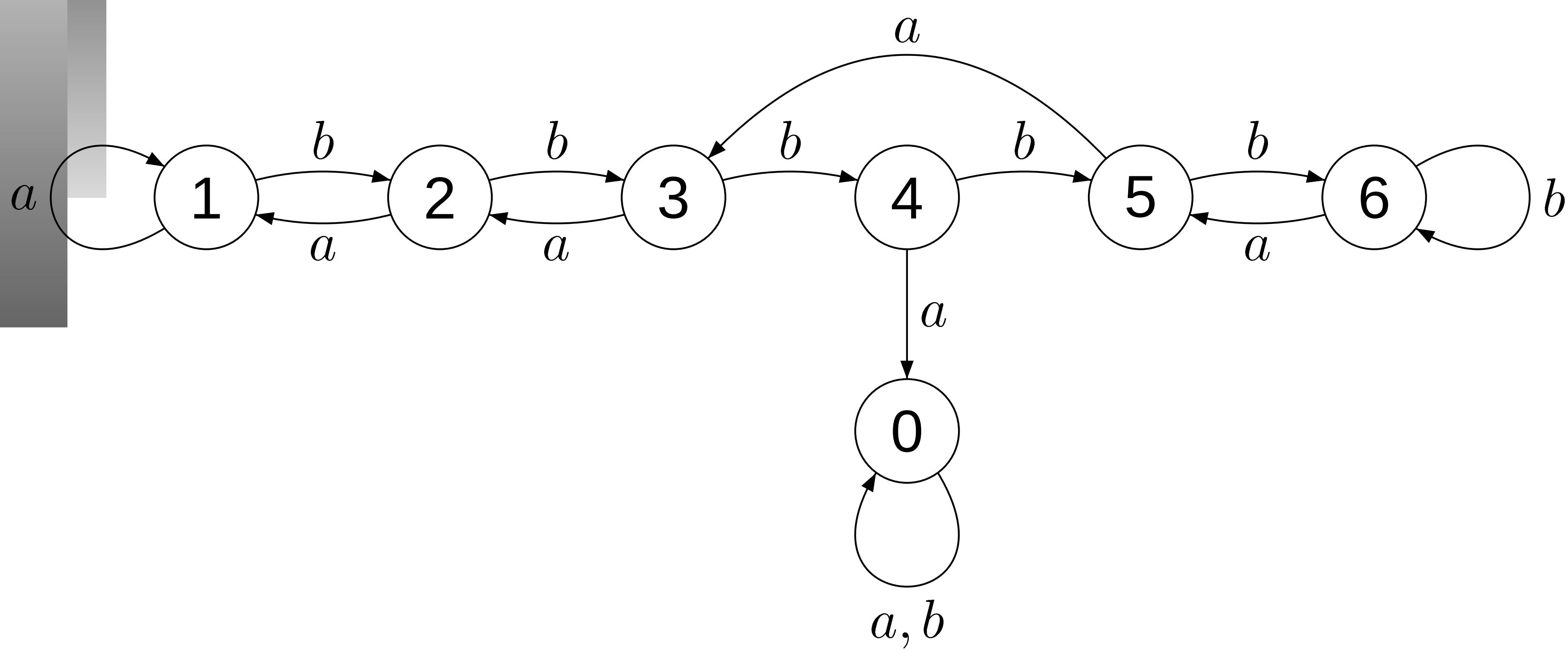

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It has 7 states and its shortest reset word is $a^4 b^3 a$ of length $7 + \left\lfloor \frac{7}{2} \right\rfloor - 2 = 8$.

Generalized Monotonicity

An equivalence relation ρ on the state set Q of a DFA $\mathscr{A} = \langle \boldsymbol{Q}, \boldsymbol{\Sigma}, \boldsymbol{\delta} \rangle$ is a *congruence* if $(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{\rho}$ implies $\big(\boldsymbol{\delta}(\boldsymbol{p}, \boldsymbol{a}), \boldsymbol{\delta}(\boldsymbol{q}, \boldsymbol{a})\big) \in \boldsymbol{\rho}$ for all $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{Q}$ and all $\boldsymbol{a} \in \boldsymbol{\Sigma}$.

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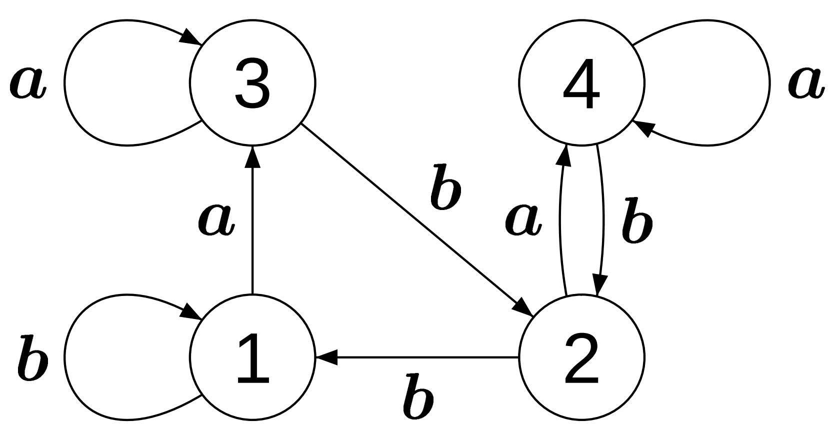

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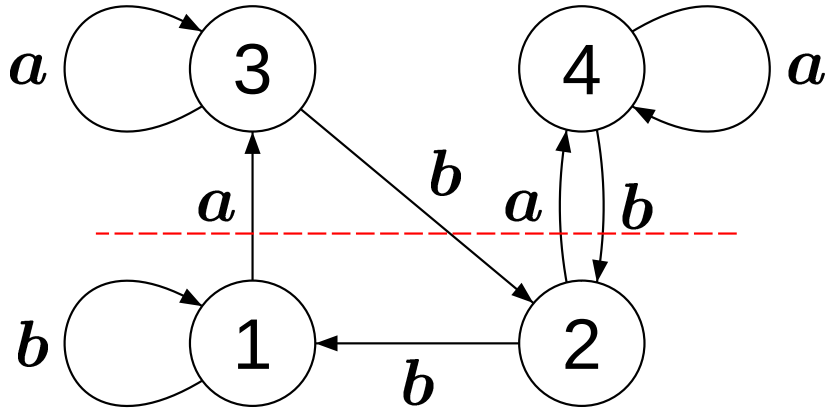

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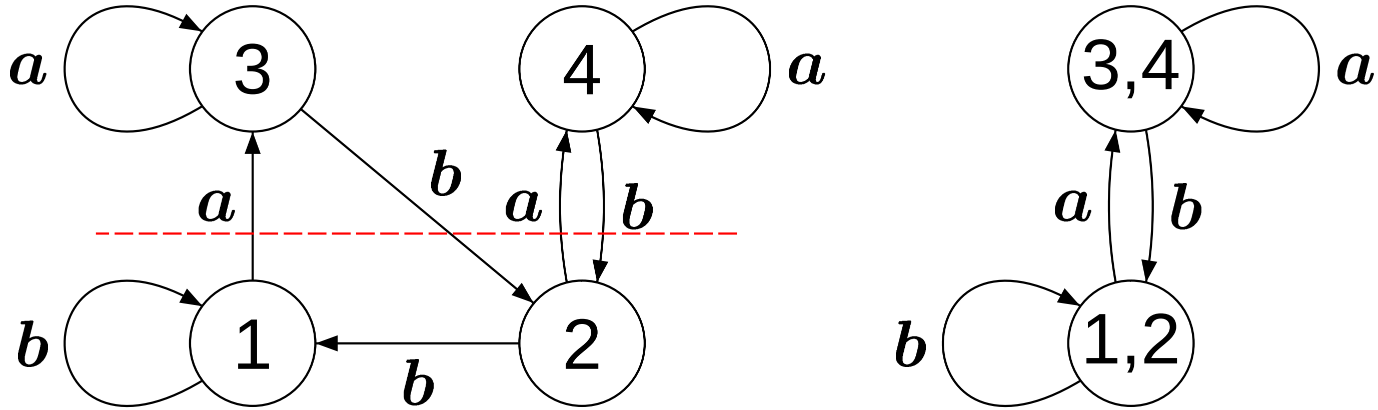

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Clearly, for ρ being universal, ρ-monotonic automata are precisely monotonic automata. On the other hand, for ρ being the equality, every DFA is ρ-monotonic.

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We call a DFA $\mathscr{A}$ *generalized monotonic of level* ℓ if it has a strictly increasing chain of congruences

$$\rho_0 \subset \rho_1 \subset \cdots \subset \rho_\ell,$$

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The automaton in the example two slides ago is a generalized monotonic automaton of level 2.

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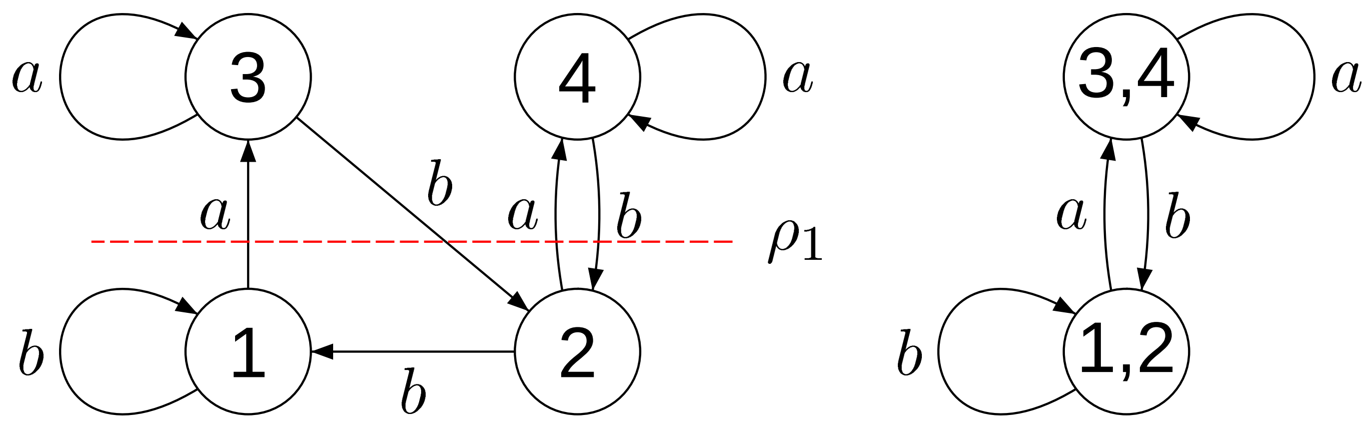

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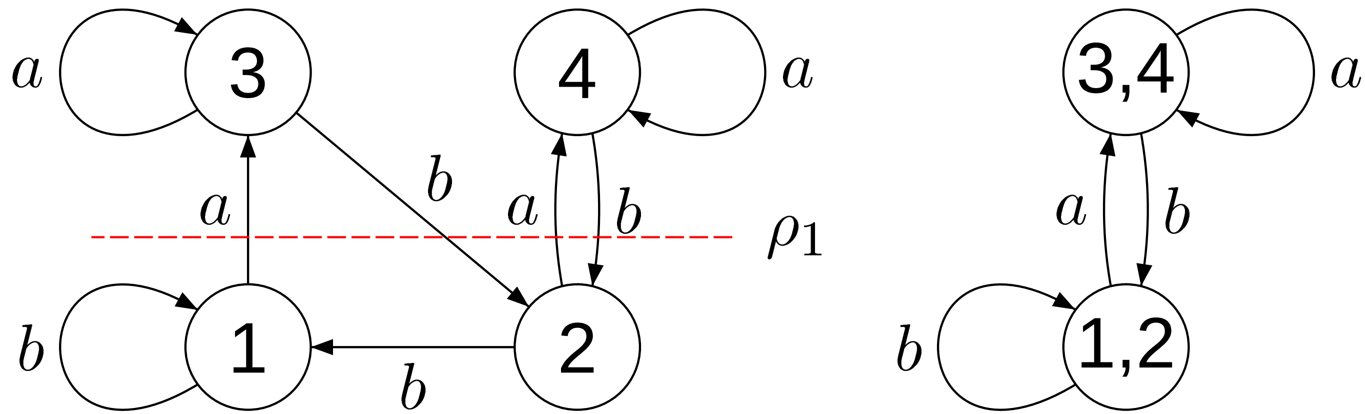

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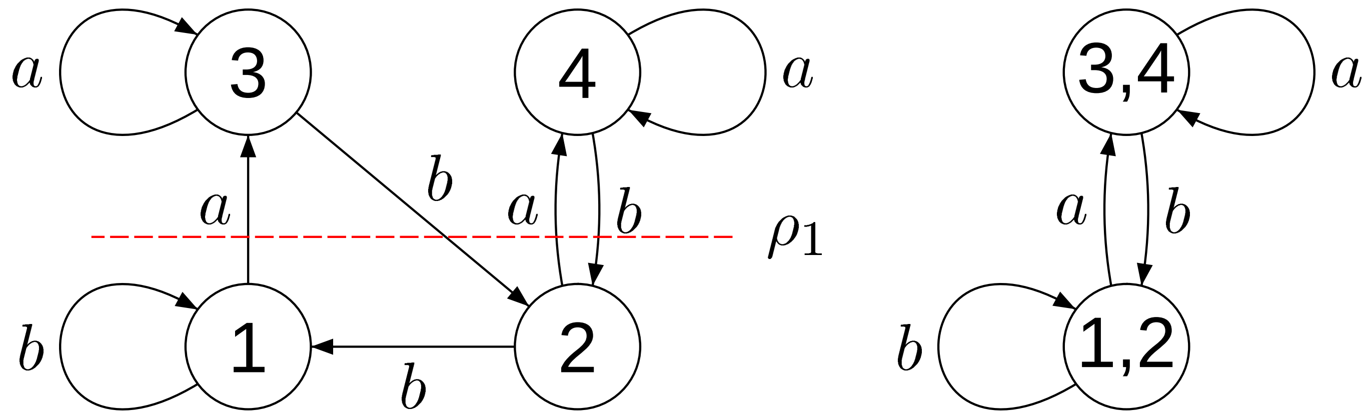

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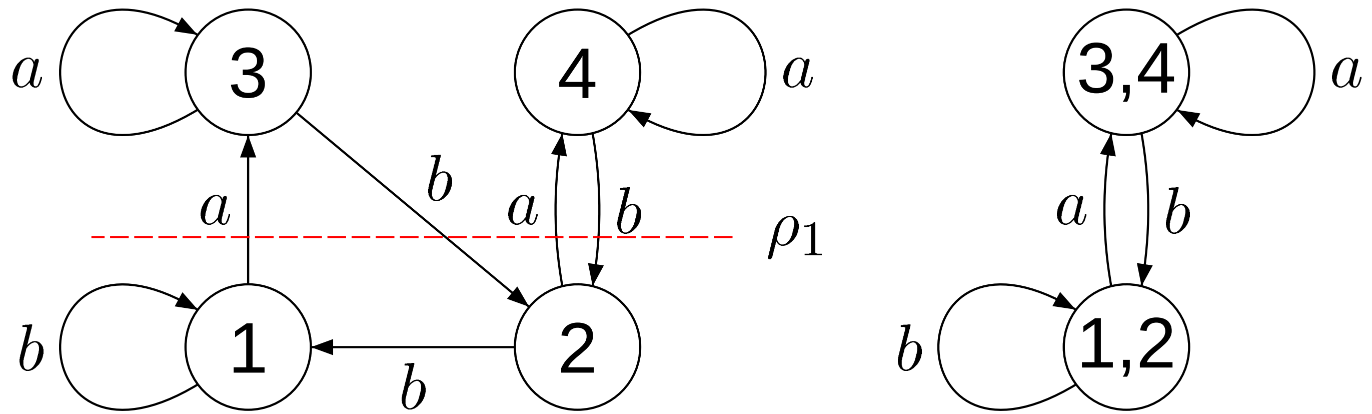

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It can be shown that the automaton is not monotonic.

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However, generalized monotonic automata are not representative for the class $\mathbf{Ap}$ from the synchronization point of view: Ananichev and $\sim$ (2005) proved that every generalized monotonic synchronizing automaton with n states has a reset word of length $\leq n - 1$.

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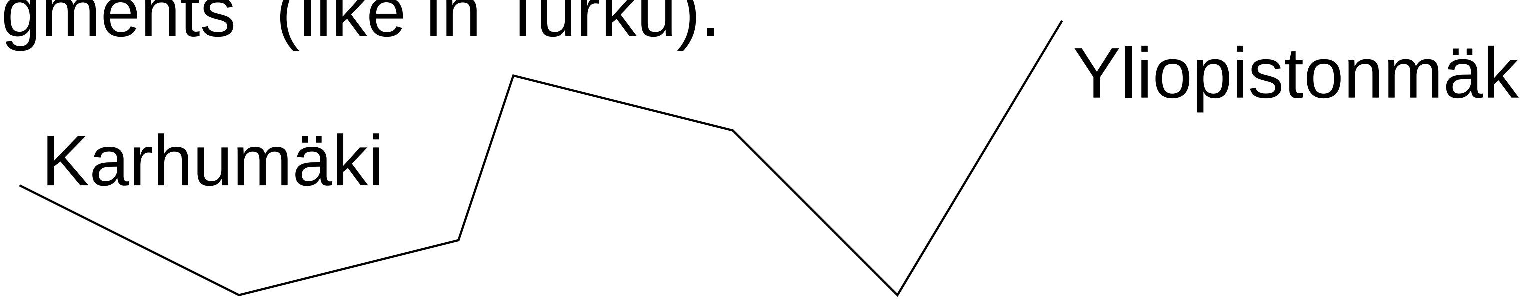
This simply means that the Hasse diagram of $\langle X, \leq \rangle$ is connected as a graph – one can walk from each point to each other via alternating uphill and downhill segments (like in Turku).



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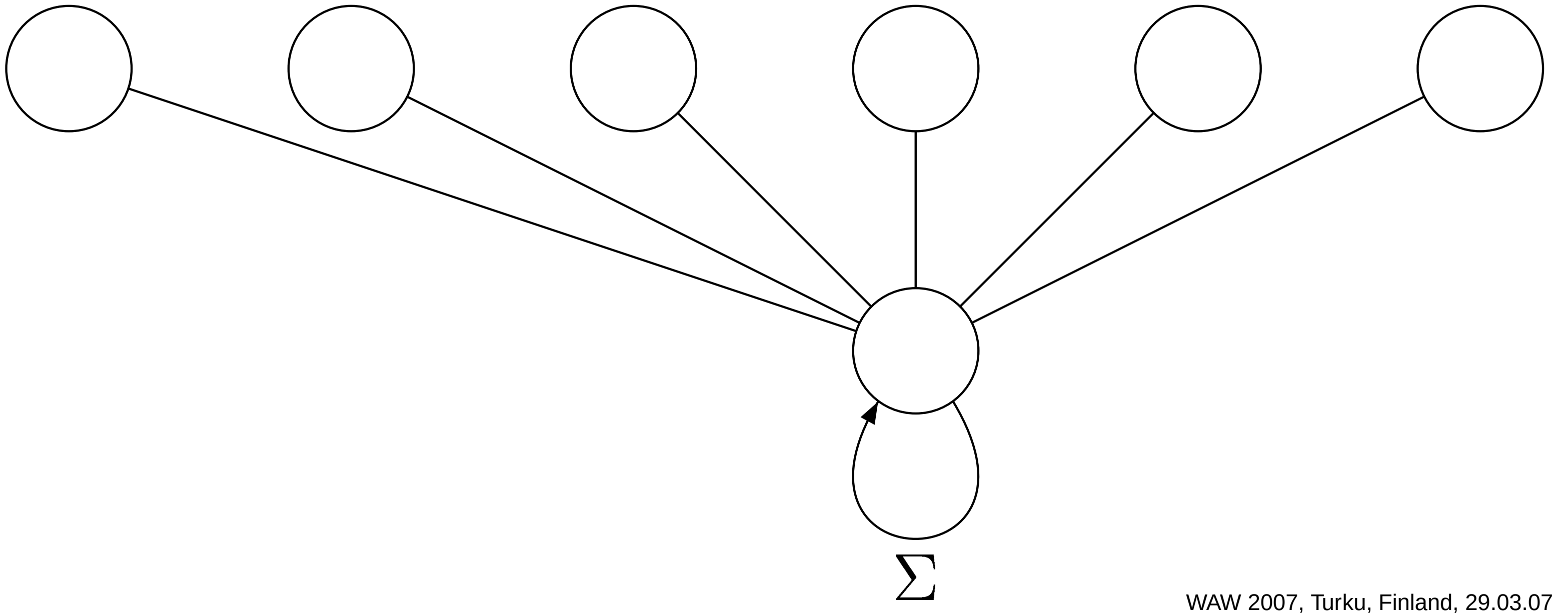

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- An arbitrary synchronizing generalized partially monotonic automaton with n states has a reset word of length $\leq \frac{n(n-1)}{2}$ and this bound is tight. (Trakhtman's upper bound for $\mathrm{SAS}(n)$ is a very special case.)

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